TRANSFORMATIONS ON THE PRODUCT OF GRASSMANN SPACES

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1. Introduction

Let $\mathcal{G}_k$ denote the set of all k-dimensional subspaces of an n-dimensional vector space. We recall that two elements of $\mathcal{G}_k$ are called *adjacent* if their intersection has dimension $k-1$. The set $\mathcal{G}_k$ is point set of a partial linear space, namely a *Grassmann space* for $1 < k < n-1$ (see Section 5) and a projective space for $k \in \{1, n-1\}$. Two adjacent subspaces are—in the language of partial linear spaces—two distinct collinear points.

W.L. Chow [4] determined all bijections of $\mathcal{G}_k$ that preserve adjacency in both directions in the year 1949. In this paper we call such a mapping, for short, an *A-transformation*. Disregarding the trivial cases $k = 1$ and $k = n-1$, every A-transformation of $\mathcal{G}_k$ is induced by a semilinear transformation $V \to V$ or (only when $k = 2n$) by a semilinear transformation of V onto its dual space V^*. There is a wealth of related results, and we refer to [2], [6], and [9] for further references.

In the present note, we aim at generalizing Chow's result to products of Grassmann spaces. However, we consider only products of the form $\mathcal{G}_k \times \mathcal{G}_{n-k}$, where $\mathcal{G}_k$ and $\mathcal{G}_{n-k}$ stem from the same vector space V. Furthermore, for a fixed k we restrict our attention to a certain subset of $\mathcal{G}_k \times \mathcal{G}_{n-k}$. This subset, say $\mathcal{G}$, is formed by all pairs of *complementary* subspaces. Our definition of an adjacency on $\mathcal{G}$ in formula (3) is motivated by the definition of lines in a product of partial linear spaces; cf. e.g. [7].

One of our main results (Theorem 2) states that Chow's theorem remains true, mutatis mutandis, for the A-transformations of $\mathcal{G}$. However, in Theorem 1 we can show even more: Let us say that two elements (S, U) and (S', U') of $\mathcal{G}$ are *close* to each other, if their Hamming distance is 1 or, said differently, if they coincide in precisely one of their components. Then the bijections of $\mathcal{G}$ onto itself which preserve this closeness relation in both directions—we call them *C-transformations* of $\mathcal{G}$—are precisely the A-transformations of $\mathcal{G}$. In this way, we obtain for $1 < k < n-1$ two characterizations of the semilinear bijections $V \to V$ and $V \to V^*$ via their action on the set $\mathcal{G}$.

Finally, we turn to the following question: What happens to our results if we replace the set $\mathcal{G}$ with the entire cartesian product $\mathcal{G}_k \times \mathcal{G}_{n-k}$? Clearly, the basic notions of adjacency and closeness remain meaningful. We describe all C-transformations of $\mathcal{G}_k \times \mathcal{G}_{n-k}$ in Theorem 3. However, in sharp contrast to Theorem 1, this is a rather trivial task, and the transformations of this kind do not deserve any interest. Then, using a result of A. Naumowicz and K. Prażmowski [7], we also determine all A-transformations of $\mathcal{G}_k \times \mathcal{G}_{n-k}$ in Theorem 4. Such mappings are closely related with collineations of the underlying partial linear space, and in general they can

be described in terms of *two* semilinear bijections, but not in terms of a *single* semilinear bijection.

Before we close this section, it is worthwhile to mention that the results from [7] could be used to describe the A-transformations of arbitrary finite products of Grassmann spaces, but this is not the topic of the present article.

2. A-transformations and C-transformations

First, we collect our basic assumptions and definitions. Throughout this paper, let V be a n-dimensional left vector space over a division ring, $2 \le n < \infty$. Suppose that $P, T \subset V$ are subspaces. They are said to be *incident* (in symbols: $P \mathrel{\mathrm{I}} T$) if $P \subset T$ or if $T \subset P$. Note that according to this definition every subspace of V is incident with 0 (the zero subspace) and with V. Furthermore, we define

$$P \sim T \;:\Leftrightarrow\; \dim P = \dim T = \dim(P \cap T) + 1, \tag{1}$$

where "$\sim$" is to be read as *adjacent*.

We put $\mathcal{G}_i$, for the set i-dimensional subspaces of V, $i = 0, 1, \dots, n$. In what follows *we fix a natural number* $k \in \{1, 2, \dots, n-1\}$ and adopt the notation

$$\mathcal{G} := \{(S, U) \in \mathcal{G}_k \times \mathcal{G}_{n-k} \mid S + U = V\}. \tag{2}$$

Hence $(S, U) \in \mathcal{G}$ means that S and U are *complementary* subspaces. On the set $\mathcal{G}$ we define two binary relations: Elements (S, U) and (S', U') of $\mathcal{G}$ are said to be *adjacent* if

$$(S = S' \text{ and } U \sim U') \text{ or } (S \sim S' \text{ and } U = U'). \tag{3}$$

By abuse of notation, this relation on $\mathcal{G}$ will also be denoted by the symbol "$\sim$". Our elements are said to be *close* to each other (in symbols: $(S, U) \approx (S', U')$) if

$$(S = S' \text{ and } U \neq U') \text{ or } (S \neq S' \text{ and } U = U'). \tag{4}$$

According to this definition, any two adjacent elements of $\mathcal{G}$ are close; the converse holds only for $k = 1$ and $k = n - 1$.

We shall establish in Lemma 6 that any two elements (S, U) and (S', U') of $\mathcal{G}$ can be connected by a finite sequence

$$(S, U) = (S_0, U_0) \sim (S_1, U_1) \sim \dots \sim (S_i, U_i) = (S', U'). \tag{5}$$

Consequently, we also have

$$(S, U) = (S_0, U_0) \approx (S_1, U_1) \approx \dots \approx (S_i, U_i) = (S', U'). \tag{6}$$

We refer to the definition of a *Plücker space* in [2, p. 199], and we point out the (inessential) difference that our relations $\sim$ and $\approx$ are anti-reflexive.

A bijection $f : \mathcal{G} \to \mathcal{G}$ is said to be an *adjacency preserving transformation* (shortly: an *A-transformation*) if f and f^{-1} transfer adjacent elements of $\mathcal{G}$ to adjacent elements; if f and f^{-1} map close elements of $\mathcal{G}$ to close elements then we say that f is a *closeness preserving transformation* (shortly: a *C-transformation*).

Example 1. For any two mappings $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ we put

$$f' \times f'' : \mathcal{G}_k \times \mathcal{G}_{n-k} \to \mathcal{G}_k \times \mathcal{G}_{n-k} : (S, U) \mapsto \big(f'(S), f''(U)\big). \tag{7}$$

Each semilinear isomorphism $l : V \to V$ induces, for $i = 1, 2, \dots, n-1$, bijections

$$G_i(l) : \mathcal{G}_i \to \mathcal{G}_i : S \mapsto l(S). \tag{8}$$

Obviously, the restriction of

$$G_k(l) \times G_{n-k}(l) \tag{9}$$

to $\mathcal{G}$ is an A-transformation and a C-transformation.

Example 2. For any two mappings $g' : \mathcal{G}_k \to \mathcal{G}_{n-k}$ and $g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k$ we put

$$g' \dot{\times} g'' : \mathcal{G}_k \times \mathcal{G}_{n-k} \to \mathcal{G}_k \times \mathcal{G}_{n-k} : (S, U) \mapsto \big(g''(U), g'(S)\big). \tag{10}$$

Let V^* denote the dual space of V. Each semilinear isomorphism $s : V \to V^*$ induces, for $i = 1, 2, \ldots, n-1$, the bijections

$$D_i(s) : \mathcal{G}_i \to \mathcal{G}_{n-i} : S \mapsto \big(s(S)\big)^\circ, \tag{11}$$

where $\big(s(S)\big)^\circ$ denotes the annihilator of $s(S)$. The restriction of

$$D_k(s) \dot{\times} D_{n-k}(s) \tag{12}$$

to $\mathcal{G}$ is an A-transformation and a C-transformation. Observe that a necessary and sufficient condition for the existence of such an isomorphism s is that the underlying division ring admits an anti-automorphism.

Example 3. Now suppose that $n = 2k$. We assume that $l : V \to V$ and $s : V \to V^*$ are semilinear isomorphisms. The restrictions of

$$G_k(l) \dot{\times} G_k(l) \ \text{ and } \ D_k(s) \times D_k(s) \tag{13}$$

to $\mathcal{G}$ both are A-transformations and C-transformations.

Example 4. Let $n = 2$ and $k = 1$. Choose an arbitrary bijection $f : \mathcal{G}_1 \to \mathcal{G}_1$. Then the restrictions of $f \times f$ and $f \dot{\times} f$ to $\mathcal{G}$ both are A-transformations and C-transformations.

We are now in a position to state our main results:

Theorem 1. *Every closeness preserving transformation of $\mathcal{G}$ is one of the mappings considered in Examples* 1–4*. Hence it is an adjacency preserving transformation.*

It is trivial that each A-transformation is a C-transformation if $k = 1$ or if $k = n-1$. In Section 4 we shall prove this statement for the general case. Thus the following statement holds true.

Theorem 2. *Every adjacency preserving transformation of $\mathcal{G}$ is one of the mappings considered in Examples* 1–4*. Hence it is a closeness preserving transformation.*

It is clear that our definitions of adjacency and closeness remain meaningful on the entire cartesian product $\mathcal{G}_k \times \mathcal{G}_{n-k}$. Also the notions of C- and A-transformation and Examples 1–4 can be carried over accordingly. However, Theorems 1 and 2 do not remain unaltered when $\mathcal{G}$ is replaced with $\mathcal{G}_k \times \mathcal{G}_{n-k}$:

Example 5. Let $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ be bijections. Then $f' \times f''$ is a C-transformation. Also, if $g' : \mathcal{G}_k \to \mathcal{G}_{n-k}$ and $g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k$ are bijections then $g' \dot{\times} g''$ is a C-transformation.

For the sake of completeness, let us state the following rather trivial result:

Theorem 3. *Every closeness preserving transformation of $\mathcal{G}_k \times \mathcal{G}_{n-k}$ is one of the mappings considered in Example* 5*.*

Example 6. If $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ are bijections which preserve adjacency in both directions then $f' \times f''$ is an A-transformation. Also, if $g' : \mathcal{G}_k \to \mathcal{G}_{n-k}$ and $g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k$ are bijections which preserve adjacency in both directions then $g' \dot{\times} g''$ is an A-transformation.

Suppose that $k = 1$ or $k = n-1$. Then it suffices to require that the mappings f', f'', g' and g'' from above are bijections in order to obtain an A-transformation of $\mathcal{G}_k \times \mathcal{G}_{n-k}$.

Provided that $1 < k < n-1$, we can apply Chow's theorem ([4, p. 38], [5, p. 81]) to describe explicitly the mappings from above.

In the first case we have $f' = G_k(l')$ or $f' = D_k(s')$ (only when $n = 2k$), and $f'' = G_{n-k}(l'')$ or $f'' = D_k(s'')$ (only when $n = 2k$).

In the second case we have $g' = D_k(s')$ or $g' = G_k(l')$ (only when $n = 2k$), and $g'' = D_{n-k}(s'')$ or $g'' = G_k(l'')$ (only when $n = 2k$).

Here $l', l'' : V \to V$ and $s', s'' : V \to V^*$ denote semilinear isomorphisms.

We shall see that the following result is a consequence of [7, Theorem 1.14]:

Theorem 4. *Every adjacency preserving transformation of $\mathcal{G}_k \times \mathcal{G}_{n-k}$ is one of the mappings considered in Example 6.*

Remark 1. Suppose that the underlying division ring of V is not of characteristic 2. Let $u \in \mathrm{GL}(V)$ be an involution. Then there exist two invariant subspaces $U_+(u)$ and $U_-(u)$ with $V = U_+(u) \oplus U_-(u)$ such that $u(x) = \pm x$ for each $x \in U_\pm(u)$. If $\dim U_+(u) = r$ then $\dim U_-(u) = n - r$, and u is called an $(r, n-r)$-*involution.*

For our fixed k let J be the set of all $(k, n-k)$-involutions. There exists a bijection

$$\gamma : J \to \mathcal{G} : u \mapsto \big(U_+(u), U_-(u)\big). \tag{14}$$

Two $(k, n-k)$-involutions u and v are said to be *adjacent* if the corresponding elements of $\mathcal{G}$ are adjacent. This holds if, and only if, the product of u and v (in any order) is a transvection $\neq 1_V$.

Now let $f : J \to J$ be a bijection which preserves adjacency in both directions. We apply Theorem 2 to the A-transformation $\gamma f \gamma^{-1} : \mathcal{G} \to \mathcal{G}$. If $n > 2$ and $n \neq 2k$ then this last mapping is given as in Example 1 or 2. This means that f can be extended to an automorphism of the group $\mathrm{GL}(V)$ as follows: To each $u \in \mathrm{GL}(V)$ we assign lul^{-1} or the contragredient of sus^{-1}, respectively.

3. Proof of Theorem 1

Our proof of Theorem 1 will be based on several lemmas and the subsequent characterization. In the case $n = 2k$ this statement is a particular case of a result in [3]. The direct analogue of Theorem 5 for buildings can be found in [1, Proposition 4.2].

Theorem 5. *Let $1 \leq k \leq n-1$. Then for any two distinct $S_1, S_2 \in \mathcal{G}_k$ the following two conditions are equivalent:*

(a) *S_1 and S_2 are adjacent,*
(b) *There exists an $S \in \mathcal{G}_k - \{S_1, S_2\}$ such that for all $U \in \mathcal{G}_{n-k}$ the condition $(S, U) \in \mathcal{G}$ implies that (S_1, U) or (S_2, U) belongs to $\mathcal{G}$.*

Proof. (a) $\Rightarrow$ (b). If S_1 and S_2 are adjacent then $S_1 \cap S_2 \in \mathcal{G}_{k-1}$ and $S_1 + S_2 \in \mathcal{G}_{k+1}$. Every $S \in \mathcal{G}_k - \{S_1, S_2\}$ satisfying the condition

$$S_1 \cap S_2 \subset S \subset S_1 + S_2 \tag{15}$$

has the required property, and at least one such S exists.

(b) $\Rightarrow$ (a). The proof of this implication will be given in several steps. First we show that

$$0 \neq W_1 \subset S_1 \text{ and } 0 \neq W_2 \subset S_2 \Rightarrow (W_1 + W_2) \cap S \neq 0. \tag{16}$$

Assume, contrary to (16), that $(W_1 + W_2) \cap S = 0$. Then there exists a complement $U \in \mathcal{G}_{n-k}$ of S containing $W_1 + W_2$. By our hypothesis, U is a complement of S_1 or S_2. This contradicts $W_1 \subset S_1$ and $W_2 \subset S_2$.

Our second assertion is

$$S_1 \cap S_2 \subset S. \tag{17}$$

This inclusion is trivial if $S_1 \cap S_2$ is zero. Otherwise, let $P \subset S_1 \cap S_2$ be an arbitrarily chosen 1-dimensional subspace. We apply (16) to $W_1 = W_2 = P$. This shows that $P \cap S \neq 0$. Hence $P \subset S$, as required.

The third step is to show that

$$\dim(S \cap S_1) = \dim(S \cap S_2) = k - 1. \tag{18}$$

By symmetry, it suffices to establish that

$$W_1 \cap (S \cap S_1) \neq 0 \tag{19}$$

for all 2-dimensional subspaces $W_1 \subset S_1$: Let us take a 1-dimensional subspace $P_2 \subset S_2$ such that $P_2 \cap S = 0$. Then (17) implies that P_2 is not contained in S_1, and for every 2-dimensional subspace $W_1 \subset S_1$ the subspace $W_1 + P_2$ is 3-dimensional. Let P_1 and Q_1 be distinct 1-dimensional subspaces contained in W_1. It follows from (16) that $P_1 + P_2$ and $Q_1 + P_2$ meet S in 1-dimensional subspaces ($\neq P_2$) which will be denoted by P and Q, respectively. As P_1 and Q_1 are distinct, so are P and Q. Therefore $P + Q$ is a 2-dimensional subspace of S. Since W_1 and $P+Q$ lie in the 3-dimensional subspace $W_1 + P_2$, they have a common 1-dimensional subspace contained in $W_1 \cap S = W_1 \cap (S \cap S_1)$. This proves (18).

Finally, we read off from (17) that

$$S_1 \cap S_2 = (S \cap S_1) \cap (S \cap S_2), \tag{20}$$

and we shall finish the proof by showing that this subspace has dimension $k - 1$. By (18) and because of $S_1 \neq S_2$, the dimension of $S_1 \cap S_2$ is either $k - 2$ or $k - 1$. Suppose, to the contrary, that

$$\dim S_1 \cap S_2 = k - 2. \tag{21}$$

Then $S \cap S_1$ and $S \cap S_2$ are distinct $(k-1)$-dimensional subspaces spanning S. There exist 1-dimensional subspaces P_1, P_2 such that

$$S_i = (S \cap S_i) + P_i \tag{22}$$

for $i = 1, 2$. We have $P_1 \neq P_2$ (otherwise (17) would give $P_1 = P_2 \subset S_1 \cap S_2 \subset S$ which is impossible), and (16) guarantees that $(P_1 + P_2) \cap S$ is a 1-dimensional subspace. Then $S_1 + S_2$ is contained in the $(k+1)$-dimensional subspace $S + P_1$ which, by the dimension formula for subspaces, contradicts (21). $\square$

Lemma 1. *If $l : V \to V$ is a semilinear isomorphism such that $G_j(l)$ is the identity for at least one $j \in \{1, 2, \dots, n-1\}$ then the same holds for all $i = 1, 2 \dots, n-1$.*

Proof. This is well known. □

Lemma 2. *Let $l_i : V \to V$ and $s_i : V \to V^*$ be semilinear isomorphisms, $i = 1, 2$. Then the following assertions hold.*

(a) *If one of the mappings $G_k(l_1) \times G_{n-k}(l_2)$ or $G_k(l_1) \dot{\times} G_k(l_2)$, when restricted to $\mathcal{G}$, is a C-transformation then $G_i(l_1) = G_i(l_2)$ for all $i = 1, 2, \dots, n-1$.*
(b) *If one of the mappings $D_k(s_1) \dot{\times} D_{n-k}(s_2)$ or $D_k(s_1) \times D_k(s_2)$, when restricted to $\mathcal{G}$, is a C-transformation then $D_i(s_1) = D_i(s_2)$ for all $i = 1, 2, \dots, n-1$.*
(c) *If $n = 2k > 2$ then none of the mappings $G_k(l_1) \times D_k(s_2)$, $D_k(s_1) \times G_k(l_2)$, $G_k(l_1) \dot{\times} D_k(s_2)$, and $D_k(s_1) \dot{\times} G_k(l_2)$ is a C-transformation, when it is restricted to $\mathcal{G}$.*

Proof. (a) Let the restriction of $G_k(l_1) \times G_{n-k}(l_2)$ to $\mathcal{G}$ be a C-transformation. Then $G_k(1_V) \times G_{n-k}(l_1^{-1}l_2)$ gives also a C-transformation. This means that for each $U \in \mathcal{G}_{n-k}$ the mapping $G_k(1_V)$ transfers the set of all k-dimensional subspaces having a non-zero intersection with U onto the set of all k-dimensional subspaces having a non-zero intersection with $l_1^{-1}l_2(U)$. However, $G_k(1_V)$ is the identity. Thus

$$l_1^{-1}l_2(U) = U, \tag{23}$$

and $G_{n-k}(l_2l_1^{-1})$ is the identity. Hence we can apply Lemma 1 to show the assertion in this particular case.

Next, let the restriction of $G_k(l_1) \dot{\times} G_k(l_2)$ to $\mathcal{G}$ be a C-transformation. Thus $n = 2k$ and the assertion follows from the previous case and

$$G_k(l_1) \dot{\times} G_k(l_2) = \big(G_k(1_V) \dot{\times} G_k(1_V)\big)\big(G_k(l_1) \times G_k(l_2)\big). \tag{24}$$

(b) can be verified similarly to (a).

(c) Assume, contrary to our hypothesis, that $G_k(l_1) \times D_k(s_2)$ gives a C-transformation. Hence $G_k(1_V) \times D_k(s_2l_1^{-1})$ is also a C-transformation and, as above, we infer that

$$D_k(s_2l_1^{-1})(U) = \big((s_2l_1^{-1})(U)\big)^\circ = U \tag{25}$$

for all $U \in \mathcal{G}_k$. Let $W \in \mathcal{G}_{k-1}$. Then there are subspaces $U_1, U_2, \dots U_{k+1} \in \mathcal{G}_k$ such that $V = \sum_{i=1}^{k+1} U_i$ and $W = \bigcap_{i=1}^{k+1} U_i$. Consequently,

$$0 = \big(s_2l_1^{-1}(V)\big)^\circ = \bigcap_{i=1}^{k+1} \big((s_2l_1^{-1})(U_i)\big)^\circ = \bigcap_{i=1}^{k+1} U_i = W \tag{26}$$

which implies $k = 1$, an absurdity.

The remaining cases can be shown in the same way. □

Let us remark that in general the assumption $n > 2$ in part (c) of this lemma cannot be dropped. Indeed, if $n = 2k = 2$ and if K is a commutative field then there exists a non-degenerate alternating bilinear form $b : V \times V \to K$. Hence $s : V \to V^* : v \mapsto b(v, \cdot)$ is a linear bijection, and $G_1(1_V) \times D_1(s)$ is the identity on $\mathcal{G}_1 \times \mathcal{G}_1$.

Lemma 3. *Let $n = 2$, whence $k = 1$. Suppose that $g' : \mathcal{G}_1 \to \mathcal{G}_1$ and $g'' : \mathcal{G}_1 \to \mathcal{G}_1$ are bijections such that one of the mappings $g' \times g''$ or $g' \dot{\times} g''$, when restricted to $\mathcal{G}$, is a C-transformation. Then $g' = g''$.*

Proof. It suffices to discuss the first case, since $1_{\mathcal{G}} \dot{\times} 1_{\mathcal{G}}$ yields a C-transformation. Now we can proceed as in the proof of Lemma 2 (a) in order to establish that the restriction of $g'^{-1}g''$ to $\mathcal{G}$ equals $1_{\mathcal{G}}$. □

We say that $\mathcal{X} \subset \mathcal{G}$ is a *C-subset* if any two distinct elements of $\mathcal{X}$ are close. (If we consider the graph of the closeness relation on $\mathcal{G}$ then a C-subset is just a clique, i.e. a complete subgraph.) A C-subset is said to be *maximal* if it is not properly contained in any C-subset. In order to describe the maximal C-subsets the following notation will be useful. If P and T are subspaces of V then we put

$$\mathcal{G}(P,T) := \{(S,U) \in \mathcal{G} \mid S \mathrel{\mathrm{I}} P \text{ and } U \mathrel{\mathrm{I}} T\}; \tag{27}$$

here we use the incidence relation from the beginning of Section 2.

Lemma 4. *The maximal C-subsets of $\mathcal{G}$ are precisely the sets $\mathcal{G}(S,V)$ with $S \in \mathcal{G}_k$, and $\mathcal{G}(V,U)$ with $U \in \mathcal{G}_{n-k}$.*

Proof. Easy verification. □

We refer to the sets described in the lemma as maximal C-subsets of *first kind* and *second kind*, respectively.

Proof of Theorem 1. (a) Let f be a C-transformation of $\mathcal{G}$. Then f and f^{-1} map maximal C-subsets to maximal C-subsets. Observe that two maximal C-subsets have a unique common element if, and only if, one of them is of first kind, say $\mathcal{G}(S,V)$, the other is of second kind, say $\mathcal{G}(V,U)$, and $(S,U) \in \mathcal{G}$.

Given $S, S' \in \mathcal{G}_k$ there exists a subspace $U \in \mathcal{G}_{n-k}$ such that $S + U = S' + U = V$. We conclude from

$$f\big(\mathcal{G}(S,V)\big) \cap f\big(\mathcal{G}(V,U)\big) = \{f\big((S,U)\big)\} \tag{28}$$

that $f\big(\mathcal{G}(S,V)\big)$ and $f\big(\mathcal{G}(V,U)\big)$ are maximal C-subsets of different kind. Likewise, $f\big(\mathcal{G}(S',V)\big)$ and $f\big(\mathcal{G}(V,U)\big)$ are of different kind, so that $f\big(\mathcal{G}(S,V)\big)$ and $f\big(\mathcal{G}(S',V)\big)$ are of the same kind.

A similar argument holds for maximal C-subsets of second kind; altogether the action of the C-transformation f on the set of maximal C-subsets is either *type preserving* or *type interchanging*.

(b) Suppose that f is type preserving. Then there exist bijections

$$g' : \mathcal{G}_k \to \mathcal{G}_k \text{ such that } f\big(\mathcal{G}(S,V)\big) = \mathcal{G}\big(g'(S),V\big) \text{ for all } S \in \mathcal{G}_k,$$
$$g'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k} \text{ such that } f\big(\mathcal{G}(V,U)\big) = \mathcal{G}\big(V,g''(U)\big) \text{ for all } U \in \mathcal{G}_{n-k};$$

thus f equals the restriction of $g' \times g''$ to $\mathcal{G}$. We distinguish four cases:

Case 1: $n = 2$. Hence $k = 1$; we deduce from Lemma 3 (a) that $g' = g''$, whence f is given as in Example 4.

Case 2: $n > 2$ and $k = 1$. Then for each $U \in \mathcal{G}_{n-1}$ the mapping g' transfers the set of all 1-dimensional subspaces contained in U to the set of all 1-dimensional subspaces contained in $g''(U)$. This means, by the fundamental theorem of projective

geometry, that there exists a semilinear isomorphism $l' : V \to V$ with $g' = G_1(l')$. Similarly, g'' is induced by a semilinear isomorphism $l'' : V \to V$.

Case 3: $n > 2$ and $k = n-1$. By symmetry, this coincides with the previous case.

Case 4: $n > 2$ and $1 < k < n-1$. Then Theorem 5 guarantees that g' and g'' are adjacency preserving in both directions; Chow's theorem ([4, p. 38], [5, p. 81]) says that g' and g'' are induced by semilinear isomorphisms. More precisely, we have $g' = G_k(l')$ with a semilinear bijection $l' : V \to V$, or $g' = D_k(s')$ with a semilinear bijection $s' : V \to V^*$ (only when $n = 2k$). A similar description holds for g''.

In cases 2–4 we infer from Lemma 2 (c) that there are only two possibilities:

Case A. $g' = G_k(l')$ and $g'' = G_{n-k}(l'')$. Now Lemma 2 (a) yields that $G_i(l') = G_i(l'')$ for all $i = 1, 2 \dots, n-1$, whence f is the restriction to $\mathcal{G}$ of $G_k(l') \times G_{n-k}(l')$; cf. Example 1.

Case B. $n = 2k$, $g' = D_k(s')$, and $g'' = D_k(s'')$. Now Lemma 2 (b) yields that $D_i(s') = D_i(s'')$ for all $i = 1, 2 \dots, n-1$, whence f is the restriction to $\mathcal{G}$ of $D_k(s') \times D_k(s')$; cf. Example 3.

(c) If f is type interchanging then there exist bijections

$$\begin{aligned} g' : \mathcal{G}_k \to \mathcal{G}_{n-k} &\text{ such that } f\big(\mathcal{G}(S,V)\big) = \mathcal{G}\big(V, g'(S)\big) \text{ for all } S \in \mathcal{G}_k, \\ g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k &\text{ such that } f\big(\mathcal{G}(V,U)\big) = \mathcal{G}(g''(U), V) \text{ for all } U \in \mathcal{G}_{n-k}; \end{aligned}$$

thus f is the restriction to $\mathcal{G}$ of $g' \dot{\times} g''$. Now we can proceed, mutatis mutandis, as in (b). So f is given as in Example 4, 2, or 3.

This completes the proof. □

4. Proof of Theorem 2

First, let us introduce the following notion: We say that $\mathcal{X} \subset \mathcal{G}$ is an *A-subset* if any two distinct elements of $\mathcal{X}$ are adjacent. (As before, such a set is just a clique of the graph given by the adjacency relation on $\mathcal{G}$.) An A-subset is said to be *maximal* if it is not properly contained in any A-subset.

If $k = 1$ or if $k = n-1$ then an A-subset is the same as a C-subset, and Lemma 4 can be applied.

Lemma 5. *Let $1 < k < n-1$. Then the maximal A-subsets of $\mathcal{G}$ are precisely the following sets:*

$$\mathcal{G}(S,T) \textit{ with } S \in \mathcal{G}_k,\ T \in \mathcal{G}_{n-k+1}, \textit{ and } S + T = V. \tag{29}$$

$$\mathcal{G}(S,T) \textit{ with } S \in \mathcal{G}_k,\ T \in \mathcal{G}_{n-k-1}, \textit{ and } S \cap T = 0. \tag{30}$$

$$\mathcal{G}(T,U) \textit{ with } T \in \mathcal{G}_{k+1},\ U \in \mathcal{G}_{n-k}, \textit{ and } T + U = V. \tag{31}$$

$$\mathcal{G}(T,U) \textit{ with } T \in \mathcal{G}_{k-1},\ U \in \mathcal{G}_{n-k}, \textit{ and } T \cap U = 0. \tag{32}$$

Proof. From [4, p. 36] we recall the following: Let $\mathcal{Y} \subset \mathcal{G}_i$, $1 < i < n-1$, be a maximal set of mutually adjacent i-dimensional subspaces of V. Then there exists a subspace $T \in \mathcal{G}_{i\pm 1}$ such that $\mathcal{Y} = \{Y \in \mathcal{G}_i \mid Y \mathrm{~I~} T\}$.

Suppose now that $\mathcal{X} \subset \mathcal{G}$ is a maximal A-subset. Clearly, there exists an element $(S,U) \in \mathcal{X}$. Since $\mathcal{X}$ is also a C-subset, we obtain that $\mathcal{X} \subset \mathcal{G}(S,V)$ or that $\mathcal{X} \subset \mathcal{G}(V,U)$.

Let $\mathcal{X} \subset \mathcal{G}(S, V)$. Then the second components of the elements of $\mathcal{X}$ are mutually adjacent elements of $\mathcal{G}_{n-k}$. Hence, by the above, they all are incident with a subset $T \in \mathcal{G}_{n-k\pm 1}$. So, due to its maximality, the set $\mathcal{X}$ is given as in (29) or (30).

Similarly, if $\mathcal{X} \subset \mathcal{G}(V, U)$ then $\mathcal{X}$ can be written as in (31) or (32).

Conversely, it is obvious that (29)–(32) define maximal A-subsets. □

We shall also make use of the following result:

Lemma 6. *Any two elements (S, U) and (S', U') of $\mathcal{G}$ can be connected by a finite sequence which is given as in formula* (5). *In particular, if $S = S'$ (or $U = U'$) then this sequence can be chosen in such a way that $S = S_0 = S_1 = \cdots = S_i$ (or $U = U_0 = U_1 = \cdots = U_i$).*

Proof. (a) First, we show the particular case when $(S, U), (S, U') \in \mathcal{G}(S, V)$ with $S \in \mathcal{G}_k$. We proceed by induction on $d := (n - k) - \dim(U \cap U')$, the case $d = 0$ being trivial.

Let $d > 0$. There exists an $(n - k - 1)$-dimensional subspace W such that $U \cap U' \subset W \subset U$. So $H := W \oplus S$ is a hyperplane of V. It cannot contain U' because of $(S, U') \in \mathcal{G}$. Thus $W' := H \cap U'$ has dimension $n - k - 1$, and there exists a 1-dimensional subspace $P' \subset U'$ with $U' = P' \oplus W'$. Consequently, $P' \not\subset H$ and we obtain

$$V = P' \oplus H = P' \oplus W \oplus S. \tag{33}$$

This means that $U'' := P' \oplus W$ is a complement of S. We have $(S, U) \sim (S, U'')$ and $(n - k) - \dim(U'' \cap U') = d - 1$. So the assertion follows from the induction hypothesis, applied to (S, U'') and (S, U').

Similarly, any two elements of $\mathcal{G}(V, U)$ with $U \in \mathcal{G}_{n-k}$ can be connected.

(b) Now we consider the general case. Let (S, U) and (S', U') be elements of $\mathcal{G}$. There exists $U'' \in \mathcal{G}_{n-k}$ which is complementary to both S and S'. Then, by (a), there exists a sequence

$$(S, U) \sim \cdots \sim (S, U'') \sim \cdots \sim (S', U'') \sim \cdots \sim (S', U') \tag{34}$$

which completes the proof. □

The statement in (a) from the above is just a particular case of a more general result on the connectedness of a *spine space*; cf. [8, Proposition 2.9].

Proof of Theorem 2. (a) We shall accomplish our task by showing that every A-transformation is a C-transformation. As has been noticed in Section 2, this is trivial if $k = 1$ or if $k = n - 1$. So let f be an A-transformation of $\mathcal{G}$ and assume that $1 < k < n - 1$.

(b) We claim that

$$f\big(\mathcal{G}(S, V)\big) \text{ is a maximal C-subset for all } S \in \mathcal{G}_k. \tag{35}$$

Let us take $T \in \mathcal{G}_{n-k+1}$ such that $\mathcal{G}(S, T)$ is a maximal A-subset. Then $f\big(\mathcal{G}(S, T)\big)$ is also a maximal A-subset. According to Lemma 5 there are four possible cases.

Case 1: $f\big(\mathcal{G}(S, T)\big)$ is given according to (29). This means $f\big(\mathcal{G}(S, T)\big) = \mathcal{G}(W, Z)$ with $W \in \mathcal{G}_k$, $Z \in \mathcal{G}_{n-k+1}$, and $W + Z = V$. We assert that in this case

$$f\big((S, U')\big) \in \mathcal{G}(W, V) \text{ for all } (S, U') \in \mathcal{G}(S, V). \tag{36}$$

In order to show this we choose an element $(S,U)\in\mathcal{G}(S,T)$. Clearly, $f\big((S,U)\big)\in\mathcal{G}(W,Z)\subset\mathcal{G}(W,V)$.

First, we suppose that (S,U) and (S,U') are adjacent. Then $P:=U\cap U'\in\mathcal{G}_{n-k-1}$. We consider the *pencil* given by P and T, i.e. the set

$$\{X\in\mathcal{G}_{n-k}\mid P\subset X\subset T\}. \tag{37}$$

It contains at least three elements; precisely one them is not complementary to S. Consequently, the intersection of the maximal A-subsets $\mathcal{G}(S,T)$ and $\mathcal{G}(S,P)$ contains more than one element. The same property holds for the intersection of the maximal A-subsets $f\big(\mathcal{G}(S,T)\big)=\mathcal{G}(W,Z)$ and $f\big(\mathcal{G}(S,P)\big)$. But this means that W is the first component of every element of $f\big(\mathcal{G}(S,P)\big)$ so that $f\big((S,U')\big)\in\mathcal{G}(W,V)$.

Next, we suppose that (S,U) and (S,U') are arbitrary. By Lemma 6, (S,U) and (S,U') can be connected by a finite sequence

$$(S,U)=(S,U_0)\sim(S,U_1)\sim\cdots\sim(S,U_i)=(S,U'), \tag{38}$$

and the arguments considered above yield that (36) holds.

Since f^{-1} is adjacency preserving, we can repeat our previous proof, with $\mathcal{G}(W,Z)$ taking over the role of $\mathcal{G}(S,T)$. Altogether, this proves

$$f\big(\mathcal{G}(S,V)\big)=\mathcal{G}(W,V). \tag{39}$$

The remaining cases, i.e., when $f\big(\mathcal{G}(S,T)\big)$ is given according to (30), (31), or (32), can be treated similarly, whence (35) holds true.

(c) Dual to (b), it can be shown that $f\big(\mathcal{G}(V,U)\big)$ is a maximal C-subset for all $U\in\mathcal{G}_{n-k}$. Thus f is a C-transformation. □

5. Proofs of Theorem 3 and Theorem 4

In the following proof we use the term *maximal C-subset* just like in Section 3.

Proof of Theorem 3. Obviously, each maximal C-subset of $\mathcal{G}_k\times\mathcal{G}_{n-k}$ has either the form $\{S\}\times\mathcal{G}_{n-k}$ with $S\in\mathcal{G}_k$ (*first kind*) or $\mathcal{G}_k\times\{U\}$ with $U\in\mathcal{G}_{n-k}$ (*second kind*). Distinct maximal C-subsets of the same kind have empty intersection, whereas maximal C-subsets of different kind have a unique common element. So every C-transformation is either type preserving, whence it can be written as $f'\times f''$, or type interchanging, whence it can be written as $g'\dot{\times}g''$. □

Let $1<k<n-1$. We shall consider below the following well known *partial linear spaces*: For each $i=2,3,\dots,n-2$ the set $\mathcal{G}_i$ is the point set of the *Grassmann space* $(\mathcal{G}_i,\mathcal{L}_i)$; the elements of its line set $\mathcal{L}_i$ are the pencils

$$\mathcal{G}_i[P,T]:=\{X\in\mathcal{G}_i\mid P\subset X\subset T\}, \tag{40}$$

where $P\in\mathcal{G}_{i-1}$, $T\in\mathcal{G}_{i+1}$, and $P\subset T$. The *Segre product* (or *product space*) of $(\mathcal{G}_k,\mathcal{L}_k)$ and $(\mathcal{G}_{n-k},\mathcal{L}_{n-k})$ is the partial linear space with point set

$$\mathcal{P}:=\mathcal{G}_k\times\mathcal{G}_{n-k} \tag{41}$$

and line set

$$\mathcal{L}:=\big\{\{S\}\times l\mid S\in\mathcal{G}_k,\ l\in\mathcal{L}_{n-k}\big\}\cup\big\{m\times\{U\}\mid m\in\mathcal{L}_k,\ U\in\mathcal{G}_{n-k}\big\}. \tag{42}$$

See [7] for further details and references.

Proof of Theorem 4.

(a) If $k = 1$ or if $k = n-1$ then the assertion follows from Theorem 3.

(b) Let $1 < k < n-1$. Given a subset $\mathcal{M} \subset \mathcal{P}$ we put

$$\mathcal{M}^{\perp} := \{(S,U) \in \mathcal{P} \mid (S,U) \perp (X,Y) \text{ for all } (X,Y) \in \mathcal{M}\}, \tag{43}$$

where the sign "$\perp$" on the right hand side means "adjacent or equal". Now let (S,U) and (S,U') be adjacent elements of $\mathcal{P}$. Then

$$\{(S,U),(S,U')\}^{\perp} = \{(S,Y) \in \mathcal{P} \mid U \cap U' \subset Y \text{ or } Y \subset U + U'\} \tag{44}$$

and

$$\{(S,U),(S,U')\}^{\perp\perp} = \{(S,Y) \in \mathcal{P} \mid U \cap U' \subset Y \subset U + U'\}. \tag{45}$$

Similarly, if (S,U) and (S',U) are adjacent elements of $\mathcal{P}$ then

$$\{(S,U),(S',U)\}^{\perp\perp} = \{(X,U) \in \mathcal{P} \mid S \cap S' \subset X \subset S + S'\}. \tag{46}$$

Next, suppose that $g : \mathcal{P} \to \mathcal{P}$ is an A-transformation. Every line of $(\mathcal{P}, \mathcal{L})$ can be written in the form (45) or (46), since it contains at least two distinct collinear points or, said differently, two adjacent elements of $\mathcal{P}$. Thus g is a collineation of the product space $(\mathcal{P}, \mathcal{L})$. By [7, Theorem 1.14], there are two possibilities:

Case 1. There exist collineations of Grassmann spaces $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ such that $g = f' \times f''$. Clearly, f' and f'' are adjacency preserving in both directions.

Case 2. There exist collineations of Grassmann spaces $g' : \mathcal{G}_k \to \mathcal{G}_{n-k}$ and $g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k$ such that $g = g' \dot{\times} g''$. As above, g' and g'' are adjacency preserving in both directions.

So g is given as in Example 6. $\square$

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